

Reconnection of Stable/Unstable Manifolds of the Harper Map

— Asymptotics-Beyond-All-Orders Approach —

Shigeru Ajisaka[†] § and Shuichi Tasaki[†]

[†] Advanced Institute for Complex Systems and Department of Applied Physics,
School of Science and Engineerings, Waseda University,
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

Abstract. The Harper map is one of the simplest chaotic systems exhibiting reconnection of invariant manifolds. The method of asymptotics beyond all orders (ABAO) is used to construct stable/unstable manifolds of the Harper map. By enlarging the neighborhood of a singularity, the perturbative solution of the unstable manifold is expressed as a Borel summable asymptotic expansion in a sector including $t = -\infty$ and is analytically continued to the other sector, where the solution acquires new terms describing heteroclinic tangles. When the parameter changes to the reconnection threshold, the stable/unstable manifolds are shown to acquire new oscillatory portion corresponding to the heteroclinic tangle after the reconnection.

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§ To whom correspondence should be addressed (g00k0056@suou.waseda.jp)

1. Introduction

Bifurcation involving chaotic motions is one of the origins of diversity in complex systems and reconnection among stable/unstable manifolds is such an example. One of the simplest systems exhibiting reconnection is the Harper map [1, 2]. For practical applications, it is helpful to have an analytical view and we analytically study the reconnection in the nearly integrable Harper map.

To study weakly perturbed stable/unstable manifolds, the conventional Melnikov perturbation method is not applicable since splitting between stable and unstable manifolds is exponentially small for small perturbation parameter σ . The method of asymptotics beyond all orders (ABAO method) is one of the useful methods dealing with such situations. The key idea is to employ the so-called inner equation, which magnifies the behavior of the solution near its singularities, and to apply the Borel transformation for investigating the inner equation and analytically continuing its solution. The idea of the inner equation was first used by Lazutkin, Schachmannski, Tabanov[3] to derive the first crossing angle between the stable and unstable manifolds of Chirikov's standard map and by Kruskal and Segur[4] to study a singular perturbation problem of ordinary differential equations. Hakim and Mallick[5] introduced the technique of Borel transformation within in this context and obtained the first crossing angle previously obtained in [6]. Tovbis, Tsuchiya and Jaffé [7] improved their method and derived analytical approximations of perturbed stable/unstable manifolds for the Hénon map. In these approaches, only the dominant part of the inner solution is taken into account. Later, Nakamura and Kushibe [10] proposed a method of systematically improving the solution of the inner equation with the aid of the Stokes multiplier and studied Chirikov's standard map. The ABAO method was applied to some higher dimensional systems as well[11, 12]. Also, Gelfreich and his collaborators[13, 14, 15, 16, 17] are studying the splitting of separatrices and related bifurcations for various systems by a slightly different but more rigorous way.

Roughly speaking, the procedure[7, 8, 9, 10] obtaining an approximate expression of the unstable manifold is summarized as follows:

- (i) Find singularities in the complex time domain of the lowest order ordinary perturbative solution of the unstable manifold.
- (ii) Magnify the neighborhood of a singularity closest to the real axis and derive an asymptotic expansion of the lowest order solution which is valid in a sector containing $t = -\infty$.
- (iii) The method of the Borel transformation is used to construct the asymptotic expansion which is valid in the other sector. Usually, there appear additional terms which are exponentially small for real time values.
- (iv) Going back to the original equation, derive equations for corrections corresponding to the exponentially small terms found in the previous step. An appropriate solution is chosen by matching its asymptotic expansion with that obtained in the previous

step.

The additional terms are determined by the singularities of the Borel-transformed solutions and, so far, only contributions from poles were considered. In this paper, with the aid of ABAO method, the reconnection of the unstable manifold is studied for the Harper map. Our analysis mainly follows the method by Nakamura and Kushibe [10], but contributions from the branch cuts of the Borel-transformed solution are also taken into account.

The Harper map depends on a real parameter k and is defined on $(v, u) \in [-\pi, \pi]^2$:

$$\begin{aligned} v(t + \sigma) - v(t) &= -\sigma \sin u(t) \\ u(t + \sigma) - u(t) &= k\sigma \sin v(t + \sigma) \end{aligned} \quad (1)$$

where $\sigma(> 0)$ is the time step and plays a role of the small parameter. Since the case of $k < 0$ is conjugate to that of $k > 0$, it is sufficient to consider the latter [2]. In the continuous time limit, where $\sigma \rightarrow 0$, the map reduces to a set of integrable differential equations:

$$\begin{aligned} v'(t) &= -\sin u(t) \\ u'(t) &= k \sin v(t) . \end{aligned} \quad (2)$$

Throughout this paper, The prime is used to indicate the differentiation with respect to time t such as $u'(t) \equiv \frac{du(t)}{dt}$. Eq.(2) admits topologically different separatrices depending on the parameter k (cf. Fig. 1) and the separatrix changes its shape when $k \rightarrow 1$. Since the solution for $k > 1$ can be obtained from that for $k < 1$ by a simple symmetry argument (cf. Appendix A), we restrict ourselves to the case of $0 < k < 1$ and consider the change of the stable/unstable manifolds for $k \rightarrow 1$. In the text, an approximation of the unstable manifold is constructed and only the result of the approximate stable manifold is given.

The rest of this paper is arranged as follows: In the next section, the lowest order solution of the Melnikov perturbation method is constructed and the inner equation is derived. In Sec. 3, the Borel resummation method is used to solve the inner equation and

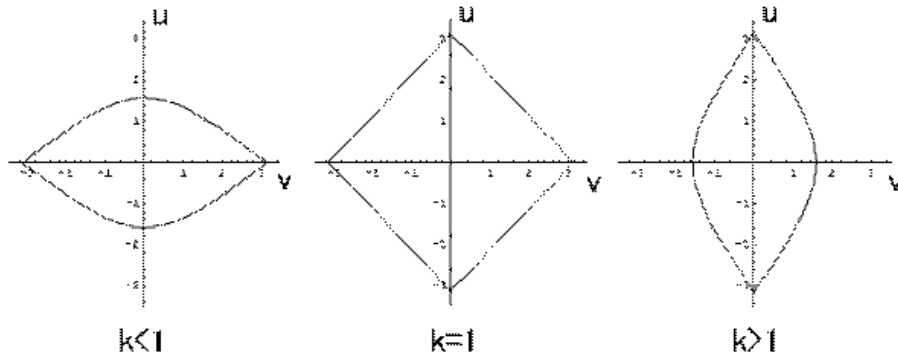


Figure 1. Separatrix of the Harper map in the continuous limit

derive the additional exponentially small terms. In Sec. 4, the solution of the original equation with the corresponding corrections is obtained and compared with numerical calculations. The reconnection of the unstable manifolds is discussed in Sec. 5. Sec. 6 is devoted to the summary.

2. Melnikov Perturbation and Inner Equation

Let v_0, u_0 be the solutions obtained by Melnikov perturbation, i.e.,

$$v_0(t) \equiv \sum_{n=0}^{\infty} \sigma^n v_{0n}(t), \quad u_0(t) \equiv \sum_{n=0}^{\infty} \sigma^n u_{0n}(t)$$

The lowest order terms satisfy (2) and the first and second order terms obey

$$\begin{aligned} v'_{01}(t) + u_{01}(t) \cos u_{00}(t) &= -\frac{1}{2} v''_{00}(t) \\ u'_{01}(t) - k v_{01}(t) \cos v_{00}(t) &= \frac{1}{2} u''_{00}(t) \end{aligned} \quad (3)$$

$$\begin{aligned} v'_{02}(t) + u_{02}(t) \cos u_{00}(t) &= -\frac{1}{2} v''_{01}(t) - \frac{1}{6} v'''_{00}(t) + \frac{u_{01}^2(t)}{2} \sin u_{00}(t) \\ u'_{02}(t) - k v_{02}(t) \cos v_{00}(t) &= \frac{1}{2} u''_{01}(t) - \frac{1}{6} u'''_{00}(t) - k \frac{v_{01}^2(t)}{2} \sin v_{00}(t) \end{aligned} \quad (4)$$

The lowest order solution of the unstable manifold with $v_0(0) = 0$ is given by

$$\begin{aligned} v_{00}(t) &= -2 \tan^{-1} \left[\sqrt{1-k} \sinh \sqrt{k}t \right] \\ u_{00}(t) &= 2 \tan^{-1} \left[\sqrt{\frac{k}{1-k}} \frac{1}{\cosh \sqrt{k}t} \right] \end{aligned} \quad (5)$$

Because of the boundary conditions $v_{0n}(t) \rightarrow 0$, $u_{0n}(t) \rightarrow 0$ for $t \rightarrow -\infty$ and $v_{0n}(0) = 0$, the first and the second order solutions are

$$v_{01}(t) = 0, \quad u_{01}(t) = \frac{1}{2} y_1(t) \quad (6)$$

$$\begin{aligned} v_{02}(t) &= -\frac{1}{24} \left[x'_1(t) + x_1(t) \left(kt - 2(1-k) \frac{\sqrt{k} \tanh \sqrt{k}t}{1 - k \tanh^2 \sqrt{k}t} \right) \right] \\ u_{02}(t) &= \frac{1}{24} \left[2y'_1(t) - y_1(t) \left(kt - 2(1-k) \frac{\sqrt{k} \tanh \sqrt{k}t}{1 - k \tanh^2 \sqrt{k}t} \right) \right] \end{aligned} \quad (7)$$

where auxiliary functions are given by

$$\begin{aligned} x_1(t) &= -2\sqrt{k(1-k)} \frac{\cosh \sqrt{k}t}{1 + (1-k) \sinh^2 \sqrt{k}t} \\ y_1(t) &= -2k\sqrt{1-k} \frac{\sinh \sqrt{k}t}{1 + (1-k) \sinh^2 \sqrt{k}t} \end{aligned} \quad (8)$$

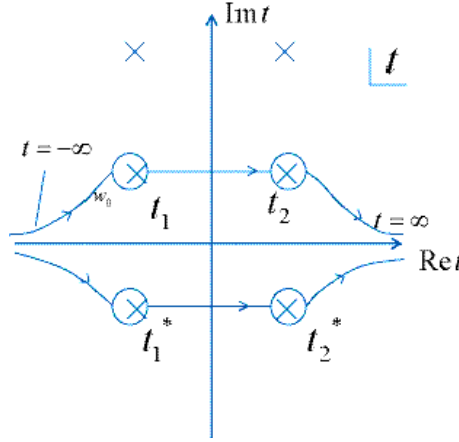


Figure 2. Singular points (crosses) of the lowest order solution v_{00} , u_{00} in the complex time domain.

We observe that the lowest order solution (5) has two sequences of singular points in the complex t -plane (see Fig. 2).

$$t = \frac{1}{\sqrt{k}} \left\{ \ln \frac{1 \pm \sqrt{k}}{\sqrt{1-k}} + \left(n + \frac{1}{2} \right) \pi i \right\} \quad (9)$$

Now we examine the behavior of perturbative solutions near two singular points in the upper half plane closest to the real axis: $t_1 \equiv \frac{1}{\sqrt{k}} \left\{ \ln \frac{1-\sqrt{k}}{\sqrt{1-k}} + \frac{1}{2} \pi i \right\}$, $t_2 \equiv \frac{1}{\sqrt{k}} \left\{ \ln \frac{1+\sqrt{k}}{\sqrt{1-k}} + \frac{1}{2} \pi i \right\}$. From now on, t_c stands for t_1 or t_2 , and the upper sign corresponds to t_1 and the lower one corresponds to t_2 .

As easily seen, the perturbative solution admits an expansion like

$$v_0(t, \sigma) - v_{00}(t) = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \frac{a_{l+1}^{(n)}}{(t - t_c)^{n-l}} \sigma^n \quad (10)$$

This indicates that the higher order terms with respect to σ has higher order poles at $t = t_c$ and that the behavior near the singularities should be carefully investigated. For this purpose, it is convenient to introduce a rescaled variable $z \equiv \frac{t-t_c}{\sigma}$ [5, 7, 8, 9, 10]. Then, one finds that $\Phi_0(z, \sigma) \equiv v_0(t_c + \sigma z, \sigma) - v_{00}(t_c + \sigma z)$ and $\Psi_0(z, \sigma) \equiv u_0(t_c + \sigma z, \sigma) - u_{00}(t_c + \sigma z)$ can be expanded into power series with respect to σ :

$$\Phi_0(z, \sigma) = \sum_{l=0}^{\infty} \Phi_{0l}(z) \sigma^l, \quad \Psi_0(z, \sigma) = \sum_{l=0}^{\infty} \Psi_{0l}(z) \sigma^l, \quad (11)$$

where (Φ_{00}, Ψ_{00}) , (Φ_{01}, Ψ_{01}) ... correspond to the most divergent terms, the next divergent terms ...

We remark that, in terms of $\Phi(z, \sigma) \equiv v(t_c + \sigma z, \sigma) - v_{00}(t_c + \sigma z)$ and $\Psi(z, \sigma) \equiv u(t_c + \sigma z, \sigma) - u_{00}(t_c + \sigma z)$, the Harper map (1) reads as

$$\begin{aligned} \Delta \Phi(z, \sigma) &= -\sigma \sin \Psi(z, \sigma) \cos u_{00}(t_c + \sigma z) \\ &\quad - \sigma \cos \Psi(z, \sigma) \sin u_{00}(t_c + \sigma z) - \Delta v_{00}(t_c + \sigma z) \end{aligned}$$

$$\begin{aligned}\Delta\Psi(z-1, \sigma) &= k\sigma \sin \Phi(z, \sigma) \cos v_{00}(t_c + \sigma z) \\ &\quad + k\sigma \cos \Phi(z, \sigma) \sin v_{00}(t_c + \sigma z) - \Delta u_{00}(t_c + \sigma(z-1))\end{aligned}\quad (12)$$

where Δ stands for the difference operator: $\Delta f(z) = f(z+1) - f(z)$, and that $\Phi_0(z, \sigma)$, $\Psi_0(z, \sigma)$ are its perturbative solutions with respect to σ . This equation will be referred to as an inner equation.

Substituting (11) into (12), one can easily obtain the equations for Φ_{0l} , Ψ_{0l} . The first two sets are as follows

$$\begin{aligned}\Delta\Phi_{00}(z) &= i \frac{e^{\mp i\Psi_{00}(z)}}{z} - i \ln \left(1 + \frac{1}{z}\right) \\ \Delta\Psi_{00}(z) &= \mp i \frac{e^{i\Phi_{00}(z+1)}}{z+1} \pm i \ln \left(1 + \frac{1}{z}\right)\end{aligned}\quad (13)$$

$$\begin{aligned}\Delta\Phi_{01}(z) &= \pm \left[i \frac{k-1}{2} + \frac{\Psi_{01}(z)}{z} \right] e^{\mp i\Psi_{00}(z)} \mp \frac{i}{2}(k-1) \\ \Delta\Psi_{01}(z) &= \left[\pm \frac{\Phi_{01}(z+1)}{z+1} - i \frac{1-k}{2} \right] e^{i\Phi_{00}(z+1)} + \frac{i}{2}(1-k)\end{aligned}\quad (14)$$

By matching the residues at $z = 0$:

$$\text{Res} \left(\frac{\Phi_{0k}(z)}{\Psi_{0k}(z)} \right) \Big|_{z=0} = \text{Res} \left(\frac{v_{0k+1}(t)}{u_{0k+1}(t)} \right) \Big|_{t=t_c}, \quad (15)$$

one obtains the solution corresponding to the expansion (10)

$$\begin{aligned}\Phi_{00}(z) &= \frac{i}{12z^2} - \frac{107i}{4320z^4} + O\left(\frac{1}{z^5}\right) \\ \Psi_{00}(z) &= \mp \left(\frac{i}{2z} - \frac{i}{24z^2} - \frac{i}{24z^3} + \frac{191i}{8640z^4} \right) + O\left(\frac{1}{z^5}\right)\end{aligned}\quad (16)$$

$$\begin{aligned}\Phi_{01}(z) &= \mp \frac{i}{24} \frac{\pm kt_c + 1}{z} + O\left(\frac{1}{z^3}\right) \\ \Psi_{01}(z) &= \frac{i(k-1)}{4} + \frac{i}{24} \frac{k(\pm t_c + 1)}{z} - \frac{i}{48} \frac{k(\pm t_c + 1)}{z^2} + O\left(\frac{1}{z^3}\right)\end{aligned}\quad (17)$$

This is the asymptotic solution of the inner equation which is valid in the sector involving $\text{Re } z = -\infty$. It is necessary to analytically continue it to another sector involving $\text{Re } z = +\infty$ in order to construct an approximation of the unstable manifold in the time domain $\text{Re } t > \text{Re } t_c$. This will be carried out in the next section.

3. Borel Transformation and Solution of Inner Equation

3.1. Borel Transformation and Analytic Continuation

As mentioned in the previous section, in order to construct an approximation of the unstable manifold in the time domain $\text{Re } t > \text{Re } t_c$, the solution of the inner equation in the sector involving $\text{Re}[z] = -\infty$ should be analytically continued to the sector involving $\text{Re}[z] = +\infty$. The analytic continuation is carried out with the aid of the Borel transformation. Let us begin with $\Phi_{00}(z)$, $\Psi_{00}(z)$.

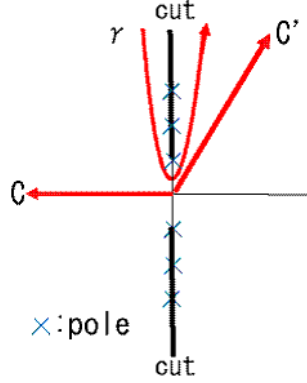


Figure 3. Integral path in p -plane

Since we are interested in the unstable manifold, we define the Borel transforms $V_0(p)$, $U_0(p)$ of $\Phi_{00}(z)$, $\Psi_{00}(z)$, respectively, as

$$\begin{aligned}\Phi_{00}(z) &\equiv L[V_0(p)](z) \equiv \int_0^{-\infty} dp e^{-pz} V_0(p) \\ \Psi_{00}(z) &\equiv L[U_0(p)](z) \equiv \int_0^{-\infty} dp e^{-pz} U_0(p)\end{aligned}\quad (18)$$

The analytic continuations of $\Phi_{00}(z)$ and $\Psi_{00}(z)$ from $\text{Re}[z] < 0$, $\text{Im}[z] < 0$ to $\text{Re}[z] > 0$ are obtained by rotating the integral path in the p -plane from C to C' (cf. Fig. 3). Other possible singularities and branch cuts of V_0 , U_0 are also shown in Fig. 3. Note that the integral path should be rotated over $\frac{\pi}{2}$ [rad] in a clockwise way so that the convergence domain rotates counterclockwise. Thus, the analytic continuation from $\text{Re}[z] < 0$, $\text{Im}[z] < 0$ to $\pi + \epsilon < \arg[z] < 2\pi + \epsilon$ is given by

$$\int_0^{-\infty} dp e^{-pz} V_0(p) \rightarrow \int_0^{-\infty e^{i(\pi/2+\epsilon)}} dp e^{-pz} V_0(p) - \int_{\gamma} dp e^{-pz} V_0(p) \quad (19)$$

where ϵ is a small positive real number.

If one can write down the Borel-transformed inner solution explicitly, one has only to change the integral path. But it is difficult and we have to deal with the inner equation and the Borel-transformed inner equation simultaneously. The procedure can be summarized as follows

(a) Write down the equation for V_0 , U_0 :

$$\begin{aligned}-i(e^{-p} - 1)V_0(p) &= 1 + \int_0^p dp \left\{ \sum_{n=1}^{\infty} \frac{(\mp i)^n U_0^{(*n)}(p)}{n!} - \frac{1 - e^{-p}}{p} \right\} \\ \pm i(1 - e^p)U_0(p) &= 1 + \int_0^p dp \left\{ \sum_{n=1}^{\infty} \frac{i^n V_0^{(*n)}(p)}{n!} - \frac{e^p - 1}{p} \right\}\end{aligned}\quad (20)$$

where $V_0^{(*n)}$ denote the n th convolution defined by

$$V_0^{(*n)}(p) = \int_0^p dx V_0(p-x) V_0^{(*n-1)}(x) \quad (21)$$

This equation indicates that the Borel transforms $V_0(p)$, $U_0(p)$ may have singularities at $p = \pm 2\pi i$ and branch cuts starting from them (cf. Fig. 3). This observation and the nonlinearity of the inner equation imply that additional terms like $-2\pi i \text{Res}(V_0(p)e^{-pz})|_{p=2\pi i n}$ and those from branch cuts may appear in $\text{Re}[z] > 0$.

(b) Observation (a) suggests the expansions

$$\begin{aligned}\Phi(z, \sigma)|_{\sigma=0} &= \sum_{n=0}^{\infty} \Phi_{n0}(z) e^{-2\pi i n z} \\ \Psi(z, \sigma)|_{\sigma=0} &= \sum_{n=0}^{\infty} \Psi_{n0}(z) e^{-2\pi i n z}\end{aligned}\quad (22)$$

where Φ_{n0} , Ψ_{n0} consist of finite polynomials of z and power series of $1/z$. Solve the equations for Φ_{n0} , Ψ_{n0} , which can be obtained by substituting (22) to (12) and comparing the same order terms with respect to $e^{-2\pi i z}$. Note that Φ_{00}, Ψ_{00} are the solutions of (16).

(c) With the aid of the formula

$$\begin{aligned}-2\pi i(-z)^{j-1}e^{-2\pi i n z} &= -\int_{\gamma} dp e^{-pz} \left[\frac{(j-1)!}{(p-2\pi i n)^j} \right] \\ \frac{-2\pi i}{z^{j+1}}e^{-2\pi i n z} &= -\int_{\gamma} dp e^{-pz} \left[\frac{(p-2\pi i n)^j \ln(p-2\pi i n)}{j!} \right]\end{aligned}$$

find $V_0(p)$, $U_0(p)$ in the upper half p -plane from the following equation

$$-\int_{\gamma} dp e^{-pz} V_0(p) = \sum_{n=1}^{\infty} \Phi_{n0}(z) e^{-2\pi i n z}, \quad -\int_{\gamma} dp e^{-pz} U_0(p) = \sum_{n=1}^{\infty} \Psi_{n0}(z) e^{-2\pi i n z}$$

In the above, we choose the branch cut of the logarithm along the positive imaginary axis as shown in Fig. 3. Note that this choice fixes the Stokes line on the negative imaginary axis in the z -plane, which corresponds to the line in the original t -plane joining t_c and its complex conjugate.

- (d) By the procedures explained so far, only the singularities in the upper half p -plane are taken into account. The contributions from the singularities in the lower half plane are determined from the fact that V_0 , U_0 is pure imaginary when z is real.
- (e) Determine V_0 , U_0 by comparing the MacLaurin expansion of the analytical guess obtained from (a)-(d) and the numerical power series solution of the Borel transformed inner equation (20).

3.2. Determination of V_0 , U_0

In this subsection, following the steps (b)-(e) explained before, V_0 , U_0 are calculated.

The equations of Φ_{10}, Ψ_{10} are given by

$$\Delta\Phi_{10}(z) = \pm \Psi_{10}(z) \frac{e^{\mp i\Psi_{00}(z)}}{z}, \quad \Delta\Psi_{10}(z) = \pm \Phi_{10}(z+1) \frac{e^{i\Phi_{00}(z+1)}}{z+1} \quad (23)$$

which have two independent solutions:

$$\begin{aligned}\Phi_{10}^{(A)} &= -\frac{1}{z} + \frac{23}{48z^3} + \dots, & \Psi_{10}^{(A)} &= \pm \left(\frac{1}{z} - \frac{1}{2z^2} - \frac{1}{48z^3} \right) + \dots \\ \Phi_{10}^{(B)} &= z + \frac{1}{12z} - \frac{17}{1080z^3} + \dots, & \Psi_{10}^{(B)} &= \pm \left(z + \frac{1}{2} - \frac{1}{540z^3} \right) + \dots\end{aligned}$$

and Φ_{10} , Ψ_{10} are their linear combinations.

$$\begin{bmatrix} \Phi_{10} \\ \Psi_{10} \end{bmatrix} = c_A \begin{bmatrix} \Phi_{10}^{(A)} \\ \Psi_{10}^{(A)} \end{bmatrix} + c_B \begin{bmatrix} \Phi_{10}^{(B)} \\ \Psi_{10}^{(B)} \end{bmatrix} \quad (24)$$

From (24), $V_0(p)$ is given by

$$\begin{aligned}- \int_{\gamma} dp e^{-pz} V_0(p) &= \sum_{n=1}^{\infty} \Phi_{n0} e^{-2\pi i n z} \\ &= \Phi_{10} e^{-2\pi i z} + \Phi_{20} e^{-4\pi i z} + \dots \\ &\approx \left[c_B \left(z + \frac{1}{12z} \right) - \frac{c_A}{z} + O\left(\frac{1}{z^3}\right) \right] e^{-2\pi i z} \quad (25)\end{aligned}$$

and U_0 can be represented in a similar way. As easily seen from (20), the coefficients c_A and c_B are imaginary number.

In order to determine the Borel transforms V_0 , U_0 from the expansion (25) by taking into account the singularities in the lower half plane, we introduce auxiliary functions $f_{lj}^{(R)}(p)$, $f_{lj}^{(I)}(p)$ satisfying the following two conditions:

- $$\begin{aligned}- \int_{\gamma} dp e^{-pz} f_{lj}^{(R)}(p) &= z^j e^{-2\pi i l z} \\ - \int_{\gamma} dp e^{-pz} f_{lj}^{(I)}(p) &= i z^j e^{-2\pi i l z}\end{aligned}$$
- $f_{lj}^{(R)}(p)$, $f_{lj}^{(I)}(p)$ is pure imaginary when p is real.

Especially, for non-negative j , $f_{1j}^{(R)}(p)$, $f_{1j}^{(I)}(p)$ are given by

$$f_{1j}^{(R)}(p) = \frac{j!}{2\pi i} \left(\frac{1}{(p - 2\pi i)^{j+1}} + \frac{(-1)^{j+1}}{(p + 2\pi i)^{j+1}} \right) \quad (26)$$

$$f_{1j}^{(I)}(p) = i \frac{j!}{2\pi i} \left(\frac{1}{(p - 2\pi i)^{j+1}} - \frac{(-1)^{j+1}}{(p + 2\pi i)^{j+1}} \right) \quad (27)$$

And $f_{1,-1}^{(I)}$ is

$$f_{1,-1}^{(I)}(p) = -\frac{1}{2\pi} [\ln(p - 2\pi i) - \ln(p + 2\pi i)] \quad (28)$$

In terms of these functions V_0 is given by

$$\begin{aligned}V_0(p) &= \frac{c_B}{i} f_{11}^{(I)}(p) + \frac{1}{i} \left(\frac{c_B}{12} - c_A \right) f_{1,-1}^{(I)}(p) \\ &= M + \frac{1}{4\pi^3} \sum_{n=0}^{\infty} \left(c_B(2n+2) - \frac{4\pi^2(\frac{c_B}{12} - c_A)}{2n+1} \right) (-1)^n \left(\frac{p}{2\pi} \right)^{2n+1} \quad (29)\end{aligned}$$

where the constant M depends on the choice of the Riemann surface of the logarithm. In a similar way, one finds

$$\begin{aligned} \pm U_0(p) &= \frac{1}{i} \left(c_B f_{11}^{(I)}(p) + c_A f_{1,-1}^{(I)}(p) + \frac{c_B}{2} f_{10}^{(I)}(p) \right) \\ &= M' + \frac{1}{4\pi^3} \sum_{n=0}^{\infty} \left(c_B(2n+2) - \frac{4\pi^2 c_A}{2n+1} \right) (-1)^n \left(\frac{p}{2\pi} \right)^{2n+1} \\ &\quad - \frac{c_B}{4\pi^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{p}{2\pi} \right)^{2n} \end{aligned} \quad (30)$$

where M' depends on the choice of the Riemann surface.

In order to check the validity of analytical guess (29) and (30) about singularities and to evaluate c_A , c_B , we derive power series solutions of (20) numerically and compare them with the power series expansions of the R.H.S. of (29) and (30). By substituting the power series expansions:

$$V_0(p) \equiv \sum_{n=0}^{\infty} a_n p^n, \quad U_0(p) \equiv \sum_{n=0}^{\infty} b_n p^n; \quad a_0 = 0, \quad b_0 = \mp i \frac{1}{2}$$

into (20) and comparing term by term, one can determine the coefficients a_n , b_n recursively. They have the following asymptotic forms.

$$\begin{aligned} -ia_{2n+1}(-1)^n(2\pi)^{2n+1} &\rightarrow 0.27893(2n+2) - \frac{0.417}{(2n+1)}, \quad \text{as } n \rightarrow \infty \\ &\equiv A_1(2n+2) - \frac{A_2}{(2n+1)} \\ \pm ib_{2n+1}(-1)^n(2\pi)^{2n+1} &\rightarrow -0.27893(2n+2) + \frac{0.417}{(2n+1)}, \quad \text{as } n \rightarrow \infty \\ -ia_{2n} &= 0, \quad \forall n \\ \pm ib_{2n}(-1)^n(2\pi)^{2n} &\rightarrow 0.87628 \equiv A_3 \end{aligned} \quad (31)$$

The numerical evaluations of A_1 , A_3 are quite robust. On the other hand, A_2 is sensitive to an error of A_1 since the coefficients of $1/(2n+1)$ are fitted after subtracting the leading terms. However, we observe that the coefficients of $1/(2n+1)$ in $a_{2n+1}(-1)^n(2\pi)^{2n+1}$ and $b_{2n+1}(-1)^n(2\pi)^{2n+1}$ are always the same. This observation suggests that the coefficients of $1/z$ in Φ_{10} , Ψ_{10} are identical, which implies

$$A_2 = \frac{c_A}{i\pi} = \frac{c_B}{24i\pi} = \frac{A_1\pi^2}{6} \quad (32)$$

The present numerical estimation gives a consistent value of 0.417 with this relation.

According to (29), (30) and (31), the asymptotic expansions of $a_{2n+1}(-1)^n(2\pi)^{2n+1}$ and $b_{2n+1}(-1)^n(2\pi)^{2n+1}$ with respect to n should have the same leading terms. This is indeed the case. Moreover, A_3/A_1 should be equal to π and we have an excellent agreement: $A_3/A_1 = 3.14158 \dots$. Therefore, we can conclude that our guess about the poles and branch cuts in the p -domain is valid. Then the constants c_A , c_B are given by $c_A = i\pi A_2$ and $c_B = i4\pi^3 A_1$.

So far, we have considered the first three dominant terms in V_0 , U_0 . From the asymptotic expansions of (25), one finds that V_0 , U_0 involve $f_{1j}^{(I)}(p)$ ($j \leq -2$) and $f_{lj}^{(I)}(p)$ ($l \geq 2$). As easily seen, one has the following rough estimations:

$$f_{lj}^{(\lambda)}(p) \sim \sum_{n=0}^{\infty} \frac{1}{n^2(2\pi l)^n} p^n \quad (j \leq -2), \quad f_{lj}^{(\lambda)}(p) \sim \sum_{n=0}^{\infty} \frac{n^j}{(2\pi l)^n} p^n \quad (j \geq -1)$$

Therefore, if $[f_{lj}^{(\lambda)}]_n$ denotes the absolute value of the coefficient of p^n , one has the following relation for large n :

$$[f_{11}^{(\lambda)}]_n \gg [f_{10}^{(\lambda)}]_n \gg [f_{1,-1}^{(\lambda)}]_n \gg [f_{20}^{(\lambda)}]_n \cdots \quad (33)$$

As a result, neglected terms are expected to be sufficiently small in the present numerical estimation.

In short, in this subsection, we have constructed the Borel transforms of Φ_0 , Ψ_0 following the procedure discussed in the previous subsection. The result is summarized as

$$\begin{aligned} V_0(p) &= 4\pi^3 A_1 f_{11}^{(I)}(p) + \pi A_2 f_{1,-1}^{(I)}(p) \\ U_0(p) &= \pm \left(4\pi^3 A_1 f_{11}^{(I)}(p) + 2\pi^3 A_1 f_{10}^{(I)}(p) + \pi A_2 f_{1,-1}^{(I)}(p) \right) \end{aligned} \quad (34)$$

3.3. Borel Transforms of Φ_{01} and Ψ_{01}

Let $\Phi_{0j}^-(z)$, $\Psi_{0j}^-(z)$ be the sum of negative powers of z in the asymptotic expansion of $\Phi_{0j}(z)$, $\Psi_{0j}(z)$ and $V_j(p)$, $U_j(p)$ be the Borel transform of Φ_{0j}^- , Ψ_{0j}^- in the sector including $z = -\infty$, then $V_j(p)$ provides the following terms in another sector including $z = +\infty$

$$-\int_{\gamma} dp e^{-pz} V_j(p) = \sum_{n=1}^{\infty} \Phi_{nj}(z) e^{-2\pi i n z} \quad (35)$$

In addition, we define $\Phi_n(z) = \sum_{j=0}^{\infty} \sigma^j \Phi_{nj}(z)$ for later use. Similar quantities Ψ_{nj} and Ψ_n are introduced for Ψ .

As the construction of V_0 , U_0 from Φ_{10} , Ψ_{10} , V_1 , U_1 can be obtained from the solutions Φ_{11} , Ψ_{11} of the following equations:

$$\begin{aligned} \Delta \Phi_{11}(z) &= \pm \frac{\Psi_{11}(z) \pm \frac{k-1}{2} z \Psi_{10}(z)}{z} e^{\mp i \Psi_{00}(z)} - i \frac{\Psi_{10}(z) \Psi_{01}(z)}{z} e^{\mp i \Psi_{00}(z)} \\ \Delta \Psi_{11}(z-1) &= \pm \frac{\Phi_{11}(z) \pm \frac{1-k}{2} z \Phi_{10}(z)}{z} e^{i \Phi_{00}(z)} \pm i \frac{\Phi_{10}(z) \Phi_{01}(z)}{z} e^{i \Phi_{00}(z)} \end{aligned} \quad (36)$$

Indeed, from this equation, asymptotic expansions of Φ_{11} , Ψ_{11} in z are obtained and, by exactly the same procedure as in the previous subsection, V_1 , U_1 are determined as (for more detail, see Appendix B)

$$\begin{aligned} V_1(p) &= \pm \left(8\pi^4 B_2(k-1) f_{12}^{(I)}(p) + 4\pi^3 (B_3(k+1) \mp B_1 k t_c) f_{11}^{(R)}(p) \right) \\ U_1(p) &= -8\pi^4 B_2(k-1) f_{12}^{(I)}(p) + 4\pi^3 (B_3(k+1) \mp B_1 k t_c) f_{11}^{(R)}(p) \\ &\quad - 4\pi^3 B_4(k-1) f_{11}^{(I)}(p) \end{aligned} \quad (37)$$

And B_1 , B_2 , B_3 and B_4 can be numerically evaluated as A_1 , A_2 and A_3 :

$$B_1 = 0.7303, \quad B_2 = 0.007399, \quad B_3 = 0.03651, \quad B_4 = 0.04649. \quad (38)$$

3.4. General Structure of Inner Solution and Stokes Multiplier

In this subsection, we study the general structure of the inner solution $\Phi_1(z) \equiv \sum_{j=0}^{\infty} \sigma^j \Phi_{1j}(z)$ $\Psi_1(z) \equiv \sum_{j=0}^{\infty} \sigma^j \Psi_{1j}(z)$, which satisfies the following equation:

$$\begin{aligned} \Delta \Phi_1(z) &= -\sigma \Psi_1(z) \cos(\Psi_0(z) + u_{00}(t_c + \sigma z)) \\ \Delta \Psi_1(z - 1) &= k\sigma \Phi_1(z) \cos(\Phi_0(z) + v_{00}(t_c + \sigma z)) \end{aligned} \quad (39)$$

For this purpose, it is convenient to introduce two solutions of the above equation admitting the expansions:

$$\begin{bmatrix} \Phi_{10}^{(A)}(z) \\ \Psi_{10}^{(A)}(z) \end{bmatrix} + \sum_{j=1}^{\infty} \sigma^j \begin{bmatrix} \Phi_{1j}^{(A)}(z) \\ \Psi_{1j}^{(A)}(z) \end{bmatrix}, \quad \begin{bmatrix} \Phi_{10}^{(B)}(z) \\ \Psi_{10}^{(B)}(z) \end{bmatrix} + \sum_{j=1}^{\infty} \sigma^j \begin{bmatrix} \Phi_{1j}^{(B)}(z) \\ \Psi_{1j}^{(B)}(z) \end{bmatrix} \quad (40)$$

where $\Phi_{1j}^{(A)}(z)$ and $\Phi_{1j}^{(B)}(z)$ are uniquely determined by posing

$$\left\{ \text{coefficient of } \frac{1}{z} \text{ in } \Phi_{1j}^{(A)}(z) \right\} = \left\{ \text{coefficient of } z \text{ in } \Phi_{1j}^{(B)}(z) \right\} = 0.$$

Then, because of the linearity of (39), one has the following expressions

$$\begin{aligned} \Phi_{10} &= \tilde{\Lambda}_0 \Phi_{10}^{(A)} + \Lambda_0 \Phi_{10}^{(B)} \\ \Phi_{11} &= \tilde{\Lambda}_0 \Phi_{11}^{(A)} + \Lambda_0 \Phi_{11}^{(B)} + \tilde{\Lambda}_1 \Phi_{10}^{(A)} + \Lambda_1 \Phi_{10}^{(B)} \\ &\dots\dots\dots \\ \Phi_{1n} &= \sum_{j=0}^n \tilde{\Lambda}_k \Phi_{1,n-k}^{(A)} + \sum_{j=0}^n \Lambda_k \Phi_{1,n-k}^{(B)} \end{aligned} \quad (41)$$

which leads to

$$\begin{aligned} \Phi_1(z) &= (\tilde{\Lambda}_0 + \sigma \tilde{\Lambda}_1 + \dots) \left(\Phi_{10}^{(A)} + \sigma \Phi_{11}^{(A)} + \dots \right) \\ &\quad + (\Lambda_0 + \sigma \Lambda_1 + \dots) \left(\Phi_{10}^{(B)} + \sigma \Phi_{11}^{(B)} + \dots \right). \end{aligned} \quad (42)$$

The coefficient $\Lambda = \sum_{n=0}^{\infty} \sigma^n \Lambda_n$ is referred to as the Stokes multiplier in [10]. Let

$\tilde{\Lambda} = \sum_{n=0}^{\infty} \sigma^n \tilde{\Lambda}_n$, then one finally has

$$\begin{bmatrix} \Phi_1 \\ \Psi_1 \end{bmatrix} = \tilde{\Lambda} \begin{bmatrix} \Phi_{10}^{(A)} + \sigma \Phi_{11}^{(A)} + \dots \\ \Psi_{10}^{(A)} + \sigma \Psi_{11}^{(A)} + \dots \end{bmatrix} + \Lambda \begin{bmatrix} \Phi_{10}^{(B)} + \sigma \Phi_{11}^{(B)} + \dots \\ \Psi_{10}^{(B)} + \sigma \Psi_{11}^{(B)} + \dots \end{bmatrix}.$$

The first two terms of the Stokes multiplier can be estimated as follows:

$$\begin{aligned} &\begin{bmatrix} -\int dp e^{-pz} \{V_0(p) + \sigma V_1(p)\} \\ -\int_{\gamma}^{\gamma} dp e^{-pz} \{U_0(p) + \sigma U_1(p)\} \end{bmatrix} \\ &= \left(\Lambda \begin{bmatrix} (\Phi_{10}^{(B)} + \Phi_{11}^{(B)} \sigma + \dots) \\ (\Psi_{10}^{(B)} + \Psi_{11}^{(B)} \sigma + \dots) \end{bmatrix} + \tilde{\Lambda} \begin{bmatrix} (\Phi_{10}^{(A)} + \Phi_{11}^{(A)} \sigma + \dots) \\ (\Psi_{10}^{(A)} + \Psi_{11}^{(A)} \sigma + \dots) \end{bmatrix} \right) e^{-2\pi iz} \\ &= \begin{bmatrix} \pm \Lambda_0 z + \sigma (\pm \Lambda_1 z \pm \frac{k-1}{6} \Lambda_0 z^2) \\ \Lambda_0 z + \sigma (\Lambda_1 z - \frac{k-1}{6} \Lambda_0 (z^2 + z)) \end{bmatrix} e^{-2\pi iz} \end{aligned} \quad (43)$$

where the contour integral is evaluated with the aid of (34) and (37). This leads to the following relations

$$\begin{aligned}\Lambda_0 &= i4\pi^3 A_1 = i48\pi^4 B_2 = i24\pi^3 B_4 \\ \Lambda_1 &= 4\pi^3 (B_3(k+1) \mp B_1 k t_c)\end{aligned}\quad (44)$$

Eq.(44) implies two linear relations among A_1 , B_2 and B_4 , which are satisfied by the present numerical estimations rather well:

$$\frac{A_1}{12\pi B_2} = 0.99998 \simeq 1 \quad \frac{A_1}{6\pi B_4} = 0.99997 \simeq 1 .$$

4. Matching of Inner and Outer Solutions

4.1. Matching at a Singular Point

In this section, solutions of the outer equations are constructed and are matched with the inner solutions. We first consider the contribution from the singularity t_1 . Corresponding to the expansion of the analytically continued inner solutions: $\Phi = \sum_{n=0}^{\infty} \Phi_n e^{-2\pi i n z}$, $\Psi = \sum_{n=0}^{\infty} \Psi_n e^{-2\pi i n z}$, the original solutions v and u acquire new terms in a sector $\text{Re } t \geq \text{Re } t_1$ of the t_1 -neighborhood:

$$\begin{aligned}v(t) &= v_0(t, \sigma) + v_1(t, \sigma) e^{-\frac{2\pi i}{\sigma} t} + v_2(t, \sigma) e^{-\frac{4\pi i}{\sigma} t} + \dots \\ u(t) &= u_0(t, \sigma) + u_1(t, \sigma) e^{-\frac{2\pi i}{\sigma} t} + u_2(t, \sigma) e^{-\frac{4\pi i}{\sigma} t} + \dots\end{aligned}\quad (45)$$

Because of $e^{-2\pi i n z} \propto \epsilon^n e^{-\frac{2\pi i n}{\sigma} t}$ with $\epsilon \equiv e^{-\frac{\pi^2}{\sigma \sqrt{k}}}$, v_n and u_n are of order of ϵ^n . By substituting (45) into (1) and comparing term by term, we obtain the equations for v_1 and u_1 :

$$\begin{aligned}v_1(t + \sigma) - v_1(t) &= -\sigma u_1(t) \cos u_0(t) \\ u_1(t + \sigma) - u_1(t) &= k\sigma v_1(t + \sigma) \cos v_0(t + \sigma)\end{aligned}\quad (46)$$

Its solution is uniquely determined by the matching condition:

$$\begin{aligned}e^{-\frac{2\pi i t}{\sigma}} \begin{bmatrix} v_1(t, \sigma) \\ u_1(t, \sigma) \end{bmatrix} \Big|_{t=t_1+\sigma z} &= e^{-2\pi i z} \left\{ \tilde{\Lambda} \begin{bmatrix} \Phi_{10}^{(A)}(z) + \sigma \Phi_{11}^{(A)}(z) + \dots \\ \Psi_{10}^{(A)}(z) + \sigma \Psi_{11}^{(A)}(z) + \dots \end{bmatrix} \right. \\ &\quad \left. + \Lambda \begin{bmatrix} \Phi_{10}^{(B)}(z) + \sigma \Phi_{11}^{(B)}(z) + \dots \\ \Psi_{10}^{(B)}(z) + \sigma \Psi_{11}^{(B)}(z) + \dots \end{bmatrix} \right\}\end{aligned}\quad (47)$$

or equivalently

$$\begin{aligned}\begin{bmatrix} v_1(t_1 + \sigma z, \sigma) \\ u_1(t_1 + \sigma z, \sigma) \end{bmatrix} &= e^{\frac{2\pi i t_1}{\sigma}} \left\{ \tilde{\Lambda} \begin{bmatrix} \Phi_{10}^{(A)}(z) + \sigma \Phi_{11}^{(A)}(z) + \dots \\ \Psi_{10}^{(A)}(z) + \sigma \Psi_{11}^{(A)}(z) + \dots \end{bmatrix} \right. \\ &\quad \left. + \Lambda \begin{bmatrix} \Phi_{10}^{(B)}(z) + \sigma \Phi_{11}^{(B)}(z) + \dots \\ \Psi_{10}^{(B)}(z) + \sigma \Psi_{11}^{(B)}(z) + \dots \end{bmatrix} \right\} .\end{aligned}\quad (48)$$

The matching condition suggests the following expansions:

$$v_1(t, \sigma) = \sigma^j e^{\frac{2\pi i t_1}{\sigma}} \sum_{n=0}^{\infty} v_{1n}(t) \sigma^n, \quad u_1(t, \sigma) = \sigma^j e^{\frac{2\pi i t_1}{\sigma}} \sum_{n=0}^{\infty} u_{1n}(t) \sigma^n \quad (49)$$

where j is an integer. Then equations for $v_{10}, u_{10} \dots$ are given by

$$\begin{aligned} v'_{10}(t) &= -u_{10}(t) \cos u_{00}(t) \\ u'_{10}(t) &= kv_{10}(t) \cos v_{00}(t) \end{aligned} \quad (50)$$

$$\begin{aligned} v'_{11}(t) + \frac{1}{2}v''_{10}(t) &= -u_{11}(t) \cos u_{00}(t) + u_{10}(t)u_{01}(t) \sin u_{00}(t) \\ u'_{11}(t) - \frac{1}{2}u''_{10}(t) &= k(v_{11}(t) \cos v_{00}(t) - v_{10}(t)v_{01}(t) \sin v_{00}(t)) \end{aligned} \quad (51)$$

First, we solve (50). Since it admits two linearly independent solutions: $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ of (8) and the following $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$:

$$\begin{aligned} x_2(t) &= \frac{-x_1(t)}{4k(1-k)} \left[\frac{(1-k)^2}{2\sqrt{k}} \left(\sinh \sqrt{k}t \cosh \sqrt{k}t + \sqrt{k}t \right) - k^{3/2} \tanh \sqrt{k}t \right] \\ y_2(t) &= \frac{y_1(t)}{4k(1-k)} \left[\frac{(1-k)^2}{2\sqrt{k}} \left(\sinh \sqrt{k}t \cosh \sqrt{k}t - \sqrt{k}t \right) + \frac{1}{\sqrt{k}} \coth \sqrt{k}t \right], \end{aligned} \quad (52)$$

one has

$$\begin{bmatrix} v_{10}(t) \\ u_{10}(t) \end{bmatrix} = a \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} + b \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} \quad (53)$$

with the matching condition

$$\begin{aligned} \sigma^j \begin{bmatrix} v_{10}(\sigma z + t_1) \\ u_{10}(\sigma z + t_1) \end{bmatrix} \Big|_{\sigma^0} &= \sigma^j \left\{ a \begin{bmatrix} x_1(\sigma z + t_1) \\ y_1(\sigma z + t_1) \end{bmatrix} + b \begin{bmatrix} x_2(\sigma z + t_1) \\ y_2(\sigma z + t_1) \end{bmatrix} \right\} \Big|_{\sigma^0} \\ &= \tilde{\Lambda}_0 \begin{bmatrix} \Phi_{10}^{(A)}(z) \\ \Psi_{10}^{(A)}(z) \end{bmatrix} + \Lambda_0 \begin{bmatrix} \Phi_{10}^{(B)}(z) \\ \Psi_{10}^{(B)}(z) \end{bmatrix} \Big|_{\text{dom.}} \\ &= \Lambda_0 \begin{bmatrix} z \\ z \end{bmatrix} \end{aligned} \quad (54)$$

where the subscript σ^0 indicates to take the terms of order σ^0 and dom. stands for the largest part for $z \rightarrow \infty$.

Because L.H.S. of (54) starts from z , one should have $j = -1$. Then, the coefficients a and b are determined by the requirement that (54) admits well defined $\sigma \rightarrow 0$ limit:

$$a = \Lambda_0 \frac{t_1(k-1)^2 + (1+k)}{4ik(k-1)}, \quad b = -\frac{2\Lambda_0}{i}.$$

Therefore, up to the 0th order in σ , we have

$$\begin{bmatrix} v_1(t) \\ u_1(t) \end{bmatrix} \approx \frac{2\Lambda_0}{i\sigma} e^{\frac{2\pi i t_1}{\sigma}} \left\{ \frac{t_1(k-1)^2 + (1+k)}{8k(k-1)} \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} - \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} \right\} \quad (55)$$

Similarly, up to the first order in σ , one obtains

$$\begin{aligned} \begin{bmatrix} v_1(t) \\ u_1(t) \end{bmatrix} &\approx \frac{2\Lambda^{(1)}}{i\sigma} e^{\frac{2\pi i t_1}{\sigma}} \left\{ \frac{t_1(k-1)^2 + (1+k)}{8k(k-1)} \begin{bmatrix} x_1(t) \\ y_1(t) + \sigma y'_1(t)/2 \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} x_2(t) \\ y_2(t) + \sigma y'_2(t)/2 \end{bmatrix} \right\} \quad (56) \end{aligned}$$

where $\Lambda^{(1)} \equiv \Lambda_0 + \sigma\Lambda_1$ is the first order Stokes multiplier.

In the t_2 -neighborhood, the unperturbed solutions v_0, u_0 acquire the following terms in a sector $\text{Re } t \geq \text{Re } t_2$:

$$\begin{bmatrix} v_1(t) \\ u_1(t) \end{bmatrix} \approx \frac{2\Lambda^{(2)}}{i\sigma} e^{\frac{2\pi i t_2}{\sigma}} \left\{ \frac{t_2(k-1)^2 - (1+k)}{8k(k-1)} \begin{bmatrix} x_1(t) \\ y_1(t) + \sigma y'_1(t)/2 \end{bmatrix} - \begin{bmatrix} x_2(t) \\ y_2(t) + \sigma y'_2(t)/2 \end{bmatrix} \right\} \quad (57)$$

where $\Lambda^{(2)}$ is the first order Stokes multiplier arising from the singularity t_2 .

4.2. Contributions from Other Singular Points

So far, contributions from t_1 and t_2 have been considered. Here, we study contributions from other singular points.

Let $t^{(n)}$ denote the singular points satisfying, $|\text{Im}[t^{(n)}]| = \frac{(2n+1)\pi^2}{2\sqrt{k}}$. Then terms arising from $t^{(n)}$ are of order ϵ^{2n+1} , where $\epsilon = e^{-\frac{\pi^2}{\sqrt{k}\sigma}}$. Since $(v_0(t), u_0(t))$ is bounded and $(v_n(t), u_n(t)) \approx e^{\sqrt{k}t}$, ($n \geq 1$) for sufficiently large t , one can neglect the terms arising from $t^{(n)}$ ($n \geq 1$) even for sufficiently large t .

On the other hand, terms arising from t_1 may produce new terms when t passes through t_2 -neighborhood, but they are negligible as they are higher order with respect to ϵ . Therefore, the overall contribution of the singular points is a simple sum of the contributions from t_1, t_2, t_1^*, t_2^* and one has the following solution of the unstable manifold in the whole time domain:

$$\begin{aligned} v_u(t) &= v_{00}(t) + \sigma^2 v_{02}(t) \\ &+ S_-(t) \text{Re} \left[\frac{4\Lambda^{(1)}}{i\sigma} e^{\frac{2\pi i t_1}{\sigma}} \left(\frac{t_1(k-1)^2 + (1+k)}{8k(k-1)} x_1(t) - x_2(t) \right) e^{-\frac{2\pi i t}{\sigma}} \right] \\ &+ S_+(t) \text{Re} \left[\frac{4\Lambda^{(2)}}{i\sigma} e^{\frac{2\pi i t_2}{\sigma}} \left(\frac{t_2(k-1)^2 - (1+k)}{8k(k-1)} x_1(t) - x_2(t) \right) e^{-\frac{2\pi i t}{\sigma}} \right] \\ u_u(t) &= u_{00}(t) + \sigma \frac{y_1(t)}{2} + \sigma^2 u_{02}(t) + \sigma^3 \left(\frac{1}{2} u'_{02}(t) - \frac{1}{24} y''_1(t) \right) \\ &+ S_-(t) \text{Re} \left[\frac{4\Lambda^{(1)}}{i\sigma} e^{\frac{2\pi i t_1}{\sigma}} \left\{ \frac{t_1(k-1)^2 + (1+k)}{8k(k-1)} \left(y_1(t) + \sigma \frac{y'_1(t)}{2} \right) \right. \right. \\ &\quad \left. \left. - \left(y_2(t) + \sigma \frac{y'_2(t)}{2} \right) \right\} e^{-\frac{2\pi i t}{\sigma}} \right] \\ &+ S_+(t) \text{Re} \left[\frac{4\Lambda^{(2)}}{i\sigma} e^{\frac{2\pi i t_2}{\sigma}} \left\{ \frac{t_2(k-1)^2 - (1+k)}{8k(k-1)} \left(y_1(t) + \sigma \frac{y'_1(t)}{2} \right) \right. \right. \\ &\quad \left. \left. - \left(y_2(t) + \sigma \frac{y'_2(t)}{2} \right) \right\} e^{-\frac{2\pi i t}{\sigma}} \right] \end{aligned} \quad (58)$$

where $S_{\pm}(t) = S(t \pm t_1^R)$ and $S(t)$ stands for the step function, and $\Lambda^{(1)}, \Lambda^{(2)}$ denote the first order Stokes multipliers from the analysis of t_1, t_2 , respectively.

4.3. Comparison with Numerical Calculation

Here we compare the analytical solution (58) obtained in the previous subsection with the numerical calculation. For this purpose, it is convenient to rewrite the solution as

$$\begin{aligned}
v_u(t) &= \tilde{v}_0(t) + 2S(t) \left[a_{11}(t) \cos \frac{2\pi t}{\sigma} + b_{11}(t) \sin \frac{2\pi t}{\sigma} \right] \\
&\quad + 2S(t + 2t_1^R) \left[a_{12}(t) \cos \frac{2\pi(t + 2t_1^R)}{\sigma} + b_{12}(t) \sin \frac{2\pi(t + 2t_1^R)}{\sigma} \right] \\
u_u(t) &= \tilde{u}_0(t) + 2S(t) \left[a_{21}(t) \cos \frac{2\pi t}{\sigma} + b_{21}(t) \sin \frac{2\pi t}{\sigma} \right] \\
&\quad + 2S(t + 2t_1^R) \left[a_{22}(t) \cos \frac{2\pi(t + 2t_1^R)}{\sigma} + b_{22}(t) \sin \frac{2\pi(t + 2t_1^R)}{\sigma} \right]
\end{aligned} \tag{59}$$

where the auxiliary functions are given by

$$\begin{aligned}
a_{11}(t) &= (K + K_1) \left(\alpha \tilde{x}_1(t) - \tilde{x}_2(t) \right) - K'_1 \beta \tilde{x}_1(t) \\
b_{11}(t) &= K'_1 \left(\alpha \tilde{x}_1(t) - \tilde{x}_2(t) \right) + (K + K_1) \beta \tilde{x}_1(t) \\
a_{12}(t) &= (K_1 - K) \left(\alpha \tilde{x}_1(t) + \tilde{x}_2(t) \right) - K'_1 \beta \tilde{x}_1(t) \\
b_{12}(t) &= -K'_1 \left(\alpha \tilde{x}_1(t) + \tilde{x}_2(t) \right) + (K - K_1) \beta \tilde{x}_1(t) \\
a_{21}(t) &= (K + K_1) \left\{ \alpha \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right) - \left(\tilde{y}_2(t) + \frac{\sigma}{2} \tilde{y}'_2(t) \right) \right\} \\
&\quad - K'_1 \beta \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right) \\
b_{21}(t) &= K'_1 \left\{ \alpha \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right) - \left(\tilde{y}_2(t) + \frac{\sigma}{2} \tilde{y}'_2(t) \right) \right\} \\
&\quad + (K + K_1) \beta \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right) \\
a_{22}(t) &= (K_1 - K) \left\{ \alpha \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right) - \left(\tilde{y}_2(t) + \frac{\sigma}{2} \tilde{y}'_2(t) \right) \right\} \\
&\quad - K'_1 \beta \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right) \\
b_{22}(t) &= -K'_1 \left\{ \alpha \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right) + \left(\tilde{y}_2(t) + \frac{\sigma}{2} \tilde{y}'_2(t) \right) \right\} \\
&\quad + (K - K_1) \beta \left(\tilde{y}_1(t) + \frac{\sigma}{2} \tilde{y}'_1(t) \right)
\end{aligned} \tag{60}$$

The functions with tilde are defined as $\tilde{f}(t) \equiv f(t + t_1^R)$ and the constants α, β, K_1, K'_1 are given by

$$K = \frac{8\pi^3 A_1}{\sigma} e^{-\frac{\pi^2}{\sigma\sqrt{k}}}, \quad K_1 = -8\pi^3 B_1 t_1^I e^{-\frac{\pi^2}{\sigma\sqrt{k}}}$$

$$\begin{aligned}
K'_1 &= 8\pi^3(B_1 k t_1^R - B_3(k+1))e^{-\frac{\pi^2}{\sigma\sqrt{k}}} \\
\alpha &= \frac{t_1^R(k-1)^2 + (1+k)}{8k(k-1)}, \quad \beta = \frac{t_1^I(k-1)}{8k}
\end{aligned} \tag{61}$$

where t_1^R and t_1^I are the real and imaginary parts of t_1 , respectively. Eq.(59) indicates that, when t exceeds $t = 0$, exponentially growing oscillatory term appears and, when t exceeds $t = -2t_1^R(> 0)$, another exponentially growing term is added. These terms should describe heteroclinic tangles.

This is indeed the case. Fig. 4, Fig. 5 shows an analytical solution of the unstable manifold from $(\pi, 0)$. One can see a large-amplitude oscillation near $(-\pi, 0)$. As shown in Fig. 4, Fig. 5, the analytical solution (59) (solid curve) well reproduces the oscillation in the numerically calculated orbit (dots). The dots show the time evolution of an ensemble consisting of 500 points. One can see the stretching of the ensemble is also well reproduced by the approximate unstable manifold. When only the leading order terms of the inner equation are retained, the agreement between the analytical and numerical results is not so good (cf. Fig. 6) and, thus, the terms in the inner solution of order σ are important.

Before closing this section, we give an approximate stable manifold $(v_s(t), u_s(t))$. By the similar approach to construct the unstable manifold, we have the similar expressions:

$$\begin{aligned}
v_s(t) &= \tilde{v}_0(t) - 2S(-t) \left[a_{11}(t) \cos \frac{2\pi t}{\sigma} + b_{11}(t) \sin \frac{2\pi t}{\sigma} \right] \\
&\quad - 2S(-t + 2t_1^R) \left[a_{12}(t) \cos \frac{2\pi(t + 2t_1^R)}{\sigma} + b_{12}(t) \sin \frac{2\pi(t + 2t_1^R)}{\sigma} \right] \\
u_s(t) &= \tilde{u}_0(t) - 2S(-t) \left[a_{21}(t) \cos \frac{2\pi t}{\sigma} + b_{21}(t) \sin \frac{2\pi t}{\sigma} \right] \\
&\quad - 2S(-t + 2t_1^R) \left[a_{22}(t) \cos \frac{2\pi(t + 2t_1^R)}{\sigma} + b_{22}(t) \sin \frac{2\pi(t + 2t_1^R)}{\sigma} \right]
\end{aligned} \tag{62}$$

5. Reconnection of Unstable Manifold

We remind that the separatrix of the continuous-limit equation is the unperturbed stable/unstable manifold. As mentioned in Introduction (cf. Fig. 1), the separatrix changes its topology depending on the parameter k (reconnection). When $0 < k < 1$, there appears a separatrix connecting $(\pi, 0)$ and $(-\pi, 0)$. As $k \rightarrow 1$, it approaches the union of two segments: the one connecting $(\pi, 0)$ with $(0, \pi)$ and the other $(0, \pi)$ with $(-\pi, 0)$. Both segments are separatrices at the parameter value $k = 1$. The corresponding topological change occurs for stable/unstable manifolds, which will be discussed in this section. As shown in the previous section, since the approximate stable and unstable manifolds are related with each other by simple symmetry, it is enough to discuss the topological change of the unstable manifold.

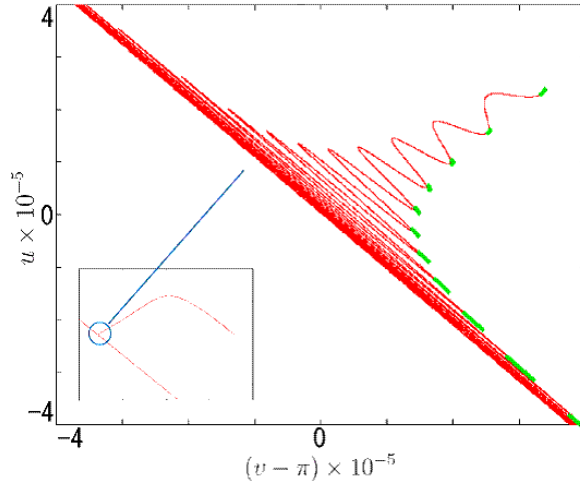


Figure 4. The analytically constructed unstable manifold (solid line) near $(-\pi, 0)$ and time evolution of the ensemble (dots). Inset shows the overall view of the analytically constructed unstable manifold. $\sigma = 0.35, k = 0.85$

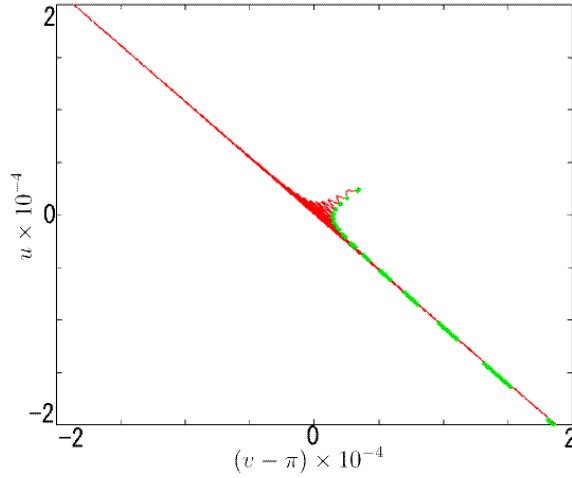


Figure 5. The analytically constructed unstable manifold (solid line) near $(-\pi, 0)$ and time evolution of the ensemble (dots). $\sigma = 0.35, k = 0.85$

Fig. 7, Fig. 8, Fig. 9, Fig. 10 show the unstable manifolds when $\sigma = 0.35$ and $k = 0.2, 0.5, 0.85, 1.0 - 10^{-9}$, respectively. The overall structure of the unstable manifold except near the fixed point $(-\pi, 0)$ is very close to that of the separatrix. Near $(-\pi, 0)$, the unstable manifold acquires an oscillatory portion and the slope of this portion becomes steeper as $k \rightarrow 1$. The behavior of the slope can be understood from the asymptotic ratio between the additional terms:

$$\lim_{t \rightarrow +\infty} \frac{u_1(t)}{v_1(t) + \pi} = -\sqrt{k} - \frac{k}{2}\sigma.$$

It is interesting to see that, when k is very close to unity (i.e., $k = 1.0 - 10^{-9}$), there appears an additional oscillatory portion in the unstable manifold near $(0, \pi)$ (cf. Fig. 5).

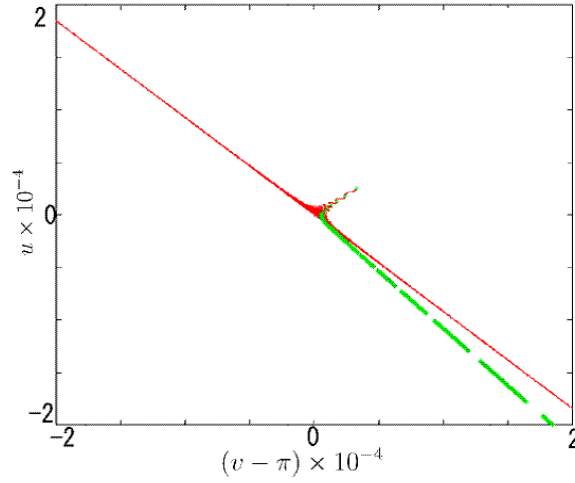


Figure 6. The analytically constructed unstable manifold (solid line) near $(-\pi, 0)$ and time evolution of the ensemble (dots). Only the leading term of the inner equation is taken into consideration. $k = 0.85, \sigma = 0.35$

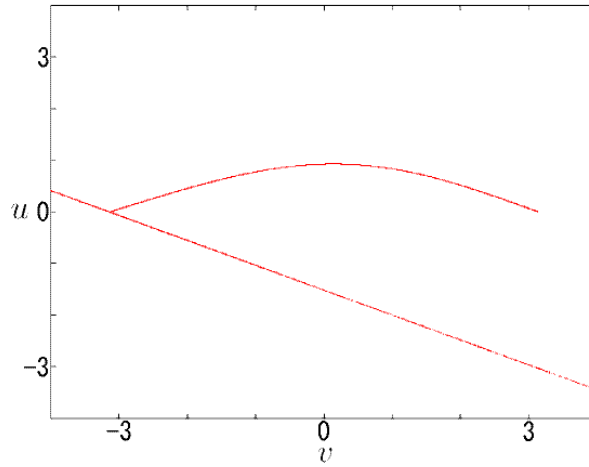


Figure 7. The analytically constructed unstable manifold. $\sigma = 0.35, k = 0.2$

As mentioned before, at $k = 1$, two segments from $(\pi, 0)$ to $(0, \pi)$ and from $(0, \pi)$ to $(-\pi, 0)$ are separatrices of the continuous-limit equation. Thus, the perturbed unstable manifold at $k = 1$ starting from $(\pi, 0)$ should have an oscillatory portion near $(0, \pi)$. The oscillation near $(0, \pi)$ of the unstable manifold shown in Fig. 5 can be considered as the precursor of the oscillation in the unstable manifold at $k = 1$.

The origin of these behaviors are summarized as follows. When k is not very close to 1, splitting term is exponentially small compared to the unperturbed solution and can be negligible for not too large t . But this term becomes dominant for sufficiently large t because $x_2, y_2 \rightarrow \infty$ as $t \rightarrow \infty$. Thus, a large-amplitude oscillation appears near $(-\pi, 0)$. This is not the case when k is very close to 1 because exponentially small terms with respect to σ are proportional to $\frac{1}{1-k}$. Therefore, the additional terms are

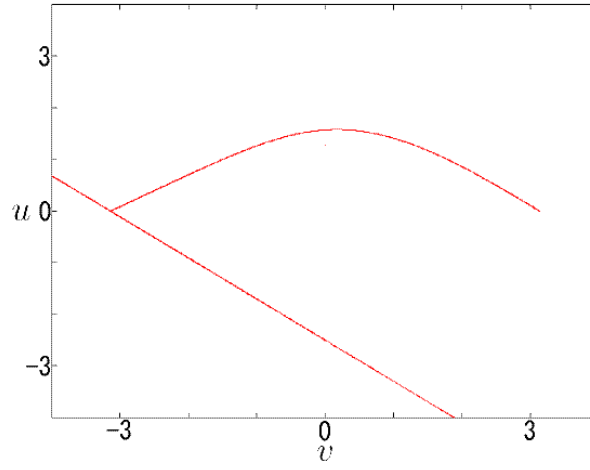


Figure 8. The analytically constructed unstable manifold. $\sigma = 0.35$, $k = 0.5$

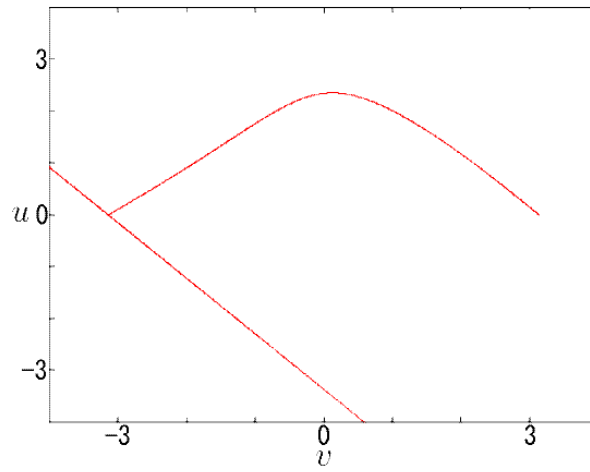


Figure 9. Analytically constructed unstable manifold. $\sigma = 0.35$, $k = 0.85$

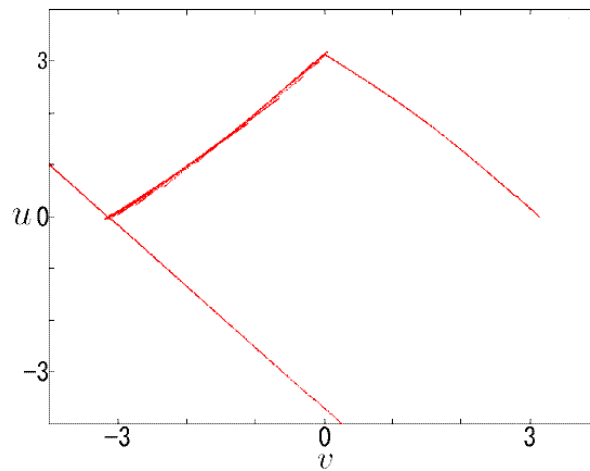


Figure 10. The analytically constructed unstable manifold. $\sigma = 0.35$, $k = 1.0 - 10^{-9}$

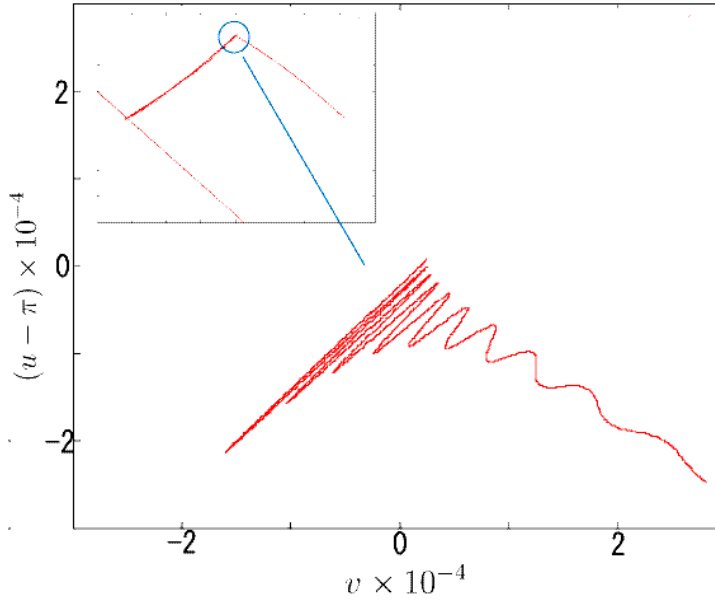


Figure 11. The analytically constructed unstable manifold and the oscillation near $(0, \pi)$. $\sigma = 0.35$, $k = 1.0 - 10^{-9}$

not too small compared to the unperturbed solution near $(0, \pi)$ and we can observe the oscillation near $(0, \pi)$. We further remark that two oscillatory portions in the unstable manifold for $k \simeq 1$ come from two singularities t_1 and t_2 in the complex time plane. In other words, the two sequences of singularities are necessary to exhibit the reconnection of stable/unstable manifolds.

6. Conclusions

With the aid of ABAO (the asymptotics beyond all orders) method, we have derived analytical approximations of the stable/unstable manifolds, which agree rather well with those obtained by the numerical iteration. When k is not close to 1, the perturbed unstable manifold starting from $(\pi, 0)$ exhibits highly oscillatory behavior near the fixed point $(-\pi, 0)$ and its overall structure far from $(-\pi, 0)$ is almost the same as that of the unperturbed unstable manifold. However, when k is very close to 1, we can observe a oscillation near $(0, \pi)$, which is considered to be the precursor of the heteroclinic tangle in the unstable manifold at $k = 1$. In this way, even when the heteroclinic tangle exists, the unstable manifold smoothly changes its topology as the change of the parameter k .

Contrary to the systems studied so far by ABAO method, the unperturbed solution of the Harper map have two sequences of singular points in the complex time plane and the interference of the contributions from them might be expected. In this paper, the approximate stable/unstable manifolds are constructed just adding the contributions from the two sequences, but they agree quite well with the numerically constructed unstable manifold. In a higher order approximation, we need to take into account this

interference. This aspect will be discussed elsewhere.

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Appendix A. Case of $k > 1$

In this section, we consider the case of $k > 1$. Put

$$\tau \equiv kt, \quad \tilde{\sigma} \equiv k\sigma, \quad \tilde{u}(\tau) \equiv v\left(-\frac{\tau}{k} + \sigma\right), \quad \tilde{v}(\tau) \equiv u\left(-\frac{\tau}{k} + \sigma\right) \quad (\text{A.1})$$

then \tilde{v}, \tilde{u} are the solution of the Harper map with parameter $1/k$:

$$\begin{aligned} \tilde{v}(\tau + \tilde{\sigma}) - \tilde{v}(\tau) &= -\tilde{\sigma} \sin \tilde{u}(\tau) \\ \tilde{u}(\tau + \tilde{\sigma}) - \tilde{u}(\tau) &= \frac{\tilde{\sigma}}{k} \sin \tilde{v}(\tau + \tilde{\sigma}) \end{aligned} \quad (\text{A.2})$$

Appendix B. Borel-transformed first order solution of inner equation

The aim of this appendix is to calculate the Borel transforms of the solutions Φ_{01}^-, Ψ_{01}^- . In order to investigate the behavior near $t = t_1$ and $t = t_2$ in a unified way, we introduce $\tilde{\Phi}_{01}(z)$ and $\tilde{\Psi}_{01}(z)$ by

$$\pm i\tilde{\Phi}_{01} \equiv \Phi_{01}^-, \quad i\tilde{\Psi}_{01} \equiv \Psi_{01}^-, \quad (\text{B.1})$$

The Borel transforms \tilde{V}_1, \tilde{U}_1 of $\tilde{\Phi}_{01}, \tilde{\Psi}_{01}$ satisfy

$$\begin{aligned} (e^{-p} - 1)\tilde{V}_1 &= \frac{k-1}{2}(g'(p) + \frac{1}{2}) + \frac{k-1}{4}(g(p) - 1) + \tilde{U}_1 * g(p) \\ (1 - e^p)\tilde{U}_1 &= \frac{k-1}{2}f'(p) + \tilde{V}_1 * f(p) \end{aligned} \quad (\text{B.2})$$

where

$$B\left[\frac{e^{i\Phi_{00}}}{z}\right] = f(p), \quad B\left[\frac{e^{\mp i\Psi_{00}}}{z}\right] = g(p)$$

The power series expansions of V_1, U_1 is defined by

$$\tilde{V}_1 = -\frac{\pm kt_c + 1}{24} + \sum_{n=1}^{\infty} \tilde{c}_n p^n, \quad \tilde{U}_1 = \frac{k(\pm t_c + 1)}{24} + \sum_{n=1}^{\infty} \tilde{d}_n p^n \quad (\text{B.3})$$

(B.2) and (B.3) gives the following form:

$$\tilde{c}_n = \pm k t_c \tilde{c}_n^{(1)} + k \tilde{c}_n^{(2)} + \tilde{c}_n^{(3)}, \quad \tilde{d}_n = \pm k t_c \tilde{d}_n^{(1)} + k \tilde{d}_n^{(2)} + \tilde{d}_n^{(3)} \quad (\text{B.4})$$

where $\tilde{c}_n^{(i)}$, $\tilde{d}_n^{(i)}$ are independent of k . By substituting (B.4) into (B.2), we numerically get the following estimation. \tilde{c}_n , \tilde{d}_n as follows:

$$\begin{aligned} \tilde{c}_{2n} &= B_2(k-1) \frac{(2n+2)(2n+1)}{(2\pi)^{2n}(-1)^n} - (\pm B_1 k t_c - B_3(k+1)) \frac{2n+1}{(2\pi)^{2n}(-1)^n} \\ \tilde{d}_{2n} &= B_2(1-k) \frac{(2n+2)(2n+1)}{(2\pi)^{2n}(-1)^n} - (\pm B_1 k t_c - B_3(k+1)) \frac{2n+1}{(2\pi)^{2n}(-1)^n} \\ \tilde{d}_{2n+1} &= -B_4(k-1) \frac{(2n+2)}{(2\pi)^{2n+1}(-1)^n} \end{aligned}$$

This estimation gives (37).

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